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THERMOELASTICITY OF A REGULARLY INHOMOGENEOUS CURVED LAYER WITH WAVY SURFACES*

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A curved inhomogeneous anisotropic layer of variable thickness is considered that has wavy surfaces. It is assumed that the elastic, thermophysical characteristics of the layer material and the shape of its upper and lower surfaces are periodic in structure with a single periodicity cell (PC). The period of the structure is here comparable in magnitude with the layer thickness, which is assumed to be much less than the other linear dimensions of the body and the radius of curvature of its middle surface.

On the basis of a general scheme for taking the average of processes in periodic media /1, 2/, a method is developed which enables a transition to be made from a spatial quasistatic thermoelasticity problem to a system of thermoelasticity equations for an average shell whose effective and thermophysical coefficients are determined from the solution of local problems in a PC. Results obtained for the static theory of elasticity in /3/ are used. The heat conduction problem is averaged to determine the temperature components occurring in the equation of motion.

The model constructed enables thermoelastic strains, stresses and the temperature distribution to be obtained in shells and plates of composite or porous materials with a different kind of reinforcement of the periodic structure (waffle, ribbed, corrugated) in reinforced and grid-like shells and plates. In the limiting case of "smooth" surfaces and a homogeneous material, the thermoelasticity equations are obtained for shallow anisotropic shells and the heat conduction equations of anisotropic shells assuming a linear temperature distribution law over the thickness.

1. The body being investigated has a periodic structure with a periodicity cell (PC) Ω_ε which is given in an orthogonal system of dimensionless coordinates $\alpha_1, \alpha_2, \gamma$ /3/ by the inequalities

$$\{0<\alpha_1<\varepsilon h,\ 0<\alpha_2<\varepsilon h,\ \gamma^-<\gamma<\gamma^+\}$$
$$\gamma^\pm=\pm\frac{\varepsilon}{2}\pm\varepsilon hF^\pm\left(\frac{\alpha_1}{\varepsilon h},\frac{\alpha_2}{\varepsilon h}\right)$$

The dimensionless small parameter ε governs the layer thickness, h characterizes the ratio of the PC dimensions to the layer thickness and is assumed to be a constant of the order of one. The functions $F^\pm$ yield the shape of the upper and lower surfaces $S^\pm$.

The physical components of the strain tensor e_{ij} and the displacement vector u_i are connected by the Cauchy relationships /3/. The equations of motion in the quasistatic formulation agree with the equilibrium equations /3/ in which the time occurs as a parameter. The stresses are connected with the strains and the temperature increment θ by the Duhamel-Neumann relationships /4, 5/

$$\sigma_{ij}=c_{ijmn}e_{mn}-c_{ijmn}\alpha_{mn}^{\theta}\theta \tag{1.1}$$

where c_{ijmn} are the coefficients of elasticity, and α_{mn}^{θ} are the temperature coefficients of linear expansion and shear. Here and henceforth, summation is over identical subscripts where $i, j, m, n=1, 2, 3;\ \mu, \nu, \beta, \delta=1, 2$.

The heat flux vector components q_i are related to the temperature by the Fourier law /4, 6/

$$q_i=-\lambda_{i\mu}\frac{1}{H_\mu}\frac{\partial\theta}{\partial\alpha_\mu}-\lambda_{i3}\frac{\partial\theta}{\partial\gamma},\quad H_\mu=A_\mu(1+k_\mu\gamma) \tag{1.2}$$

where λ_{ij} are thermal conductivities, H_μ are Lame coefficients, A_μ are coefficients of the first quadratic form, k_μ are the principal middle surface curvature, and $\alpha=(\alpha_1, \alpha_2)$.

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Without taking account of thermoelastic energy dissipation, the heat balance equation under deformation can be written in the form /4, 6/

$$-f + c_v \frac{\partial\theta}{\partial t} = -\frac{1}{H_1H_2}\left[\frac{\partial (H_2q_1)}{\partial\alpha_1} + \frac{\partial (H_1q_2)}{\partial\alpha_2} + \frac{\partial (H_1H_2q_3)}{\partial\gamma}\right] \tag{1.3}$$

(f is the density of internal heat sources, and c_v is the volume specific heat).

The force and thermal conditions

$$\sigma_{i\mu}H_\mu^{-1}n_\mu^\pm + \sigma_{i3}n_3^\pm = \pm p_i^\pm \quad (\gamma = \gamma^\pm) \tag{1.4}$$

$$q_\mu H_\mu^{-1}n_\mu^\pm + q_3n_3^\pm = \pm a_S^\pm\theta \mp g_S^\pm \quad (\gamma = \gamma^\pm) \tag{1.5}$$

$$\left(\mathbf{n}^\pm = \left\{-\frac{\partial\gamma^\pm}{\partial\alpha_1}, -\frac{\partial\gamma^\pm}{\partial\alpha_2}, 1\right\}\left[1 + \frac{1}{H_1^2}\left(\frac{\partial\gamma^\pm}{\partial\alpha_1}\right)^2 + \frac{1}{H_2^2}\left(\frac{\partial\gamma^\pm}{\partial\alpha_2}\right)^2\right]^{-1/2}\right)$$

are satisfied on the surfaces $S^\pm (\gamma = \gamma^\pm)$ where $p_i^\pm$ are external load components, $a_S^\pm$ are heat transfer coefficients, $g_S^\pm$ are external heat fluxes, and $n_i^\pm$ are components of the unit normals $\mathbf{n}^\pm$ to the surfaces $S^\pm$.

In the case when convective heat transfer occurs on the surfaces $S^\pm$, we set

$$g_S^\pm = a_S^\pm\theta_S^\pm \tag{1.6}$$

in (1.5) ($\theta_S^\pm$ is the temperature of the environment).

2. We introduce new coordinates and write the PC Ω in the form

$$\{0 < y_1 < 1,\ \ 0 < y_2 < 1,\ z^- < z < z^+\},\ z^\pm = \pm{}^1\!/_2 \pm hF^\pm(y)$$

$$\left(y_1 = \frac{\alpha_1}{\varepsilon h},\ y_2 = \frac{\alpha_2}{\varepsilon h},\ z = \frac{\gamma}{\varepsilon},\ y = (y_1, y_2)\right)$$

Following the formulation of the problem, we consider the elastic and thermophysical characteristics of the material $c_{ijmn}(y, z)$, $\alpha_{mn}^\theta(y, z)$, $\lambda_{ij}(y, z)$, $c_v(y, z)$, $a_S^\pm(y)$ to be periodic functions of y_1, y_2 with the PC Ω. The functions $f(\alpha, t, y, z)$, $g_S^\pm(\alpha, t, y)$, as well as the external volume $P(\alpha, t, y, z)$ and surface $p_i^\pm(\alpha, t, y)$ forces, may depend on both α_1, α_2, t and on y_1, y_2 with the same PC.

We seek the solution in the form of the asymptotic expansions /1, 2/

$$u_i = u_i^{(0)}(\alpha, t) + \varepsilon u_i^{(1)} + O(\varepsilon^2) \tag{2.1}$$

$$\theta = \theta_1 + z\theta_2,\ \theta_\nu = \theta_\nu^{(0)}(\alpha, t) + \varepsilon\theta_\nu^{(1)} + O(\varepsilon^2) \tag{2.2}$$

Here $u_i^{(l)}(\alpha, t, y, z)$, $\theta_\nu^{(l)}(\alpha, t, y, z)$ $(l = 1, 2, \ldots)$ are periodic functions of y_1, y_2 with the PC Ω. The principal term of the expansion (2.2) corresponds to the linear temperature distribution law over the thickness, which is taken in deriving the heat conduction equations for thin plates and shells /4, 7/.

On the basis of expansion (2.1), the problem is averaged in /3/ in the absence of the thermal component in (1.1) and it is shown that the following expansion in ε holds for the stresses:

$$\sigma_{ij} = \varepsilon\sigma_{ij}^{(1)} + O(\varepsilon^2) \tag{2.3}$$

The averaged equations

$$\begin{aligned} &\varepsilon^2\mathbf{B}_\nu\langle\sigma_{\beta\delta}^{(1)}\rangle + p_\nu + \varepsilon\langle P_\nu\rangle = 0 \\ &\frac{1}{A_1A_2}\frac{\partial}{\partial\alpha_\mu}(\varepsilon^3\mathbf{B}_\mu\langle z\sigma_{\beta\delta}^{(1)}\rangle + p_\mu^* + \varepsilon\langle\gamma P_\mu\rangle) - \varepsilon^2k_1\langle\sigma_{11}^{(1)}\rangle - \\ &\quad \varepsilon^2k_2\langle\sigma_{22}^{(1)}\rangle + p_3 + \varepsilon\langle P_3\rangle = 0 \end{aligned} \tag{2.4}$$

are obtained for the principal terms of the expansion (2.3).

Here (V is the volume of Ω)

$$p_i = \frac{1}{V}\int_0^1\int_0^1(\omega^+p_i^+ + \omega^-p_i^-)\,dy_1\,dy_2 \tag{2.5}$$

$$p_\mu^* = \frac{1}{V}\int_0^1\int_0^1(\gamma^+\omega^+p_\mu^+ + \gamma^-\omega^-p_\mu^-)\,dy_1\,dy_2$$

$$\omega^\pm = \left[1 + \frac{1}{A_1^2}\left(\frac{\partial F^\pm}{\partial y_1}\right)^2 + \frac{1}{A_2^2}\left(\frac{\partial F^\pm}{\partial y_2}\right)^2\right]^{1/2} \tag{2.6}$$

$$\langle\varphi\rangle = \frac{1}{V}\int_{\Omega} \varphi\, dy_1\, dy_2\, dz \tag{2.7}$$

$$\mathbf{B}_1(\varphi_{\mu\nu}) = \frac{1}{A_1A_2}\left[\frac{\partial(A_2\varphi_{11})}{\partial\alpha_1} + \frac{\partial(A_1\varphi_{21})}{\partial\alpha_2} + \frac{\partial A_1}{\partial\alpha_2}\varphi_{12} - \frac{\partial A_2}{\partial\alpha_1}\varphi_{22}\right] (1 \leftrightarrow 2) \tag{2.8}$$

The averaged stresses and their moments in (2.4) are connected with the strains of the layer middle surface by the following elasticity relationships /3/

$$\langle z^l\sigma_{\beta\delta}^{(1)}\rangle = \langle z^l b_{\beta\delta}^{\mu\nu}\rangle\,\omega_{\mu\nu} + \langle z^l b_{\beta\delta}^{*\mu\nu}\rangle\,\tau_{\mu\nu} \quad (l = 0, 1) \tag{2.9}$$

$$\omega_{11} = \frac{1}{A_1}\frac{\partial v_1}{\partial\alpha_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial\alpha_2}v_2 + \frac{1}{\varepsilon}k_1 w \quad (1 \leftrightarrow 2) \tag{2.10}$$

$$\omega_{12} = \omega_{21} = \frac{1}{2}\left[\frac{A_1}{A_2}\frac{\partial}{\partial\alpha_2}\left(\frac{v_1}{A_1}\right) + \frac{A_2}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{v_2}{A_2}\right)\right], \quad \omega_{3\mu} = \frac{1}{A_\mu}\frac{\partial u_3^{(1)}}{\partial\alpha_\mu}$$

$$\tau_{11} = -\frac{1}{A_1}\frac{\partial}{\partial\alpha_1}\left(\frac{1}{A_1}\frac{\partial w}{\partial\alpha_1}\right) - \frac{1}{A_1A_2^2}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_2} \quad (1 \leftrightarrow 2)$$

$$\tau_{12} = \tau_{21} = -\frac{1}{A_1A_2}\left(\frac{\partial^2 w}{\partial\alpha_1\partial\alpha_2} - \frac{1}{A_1}\frac{\partial A_1}{\partial\alpha_2}\frac{\partial w}{\partial\alpha_1} - \frac{1}{A_2}\frac{\partial A_2}{\partial\alpha_1}\frac{\partial w}{\partial\alpha_2}\right)$$

The functions v_μ, w from (2.10) govern the principle terms in the expansion of the displacement vector (2.1):

$$u_\mu^{(0)} = 0, \quad u_3^{(0)} = w(\alpha, t), \quad u_\mu^{(1)} = v_\mu(\alpha, t) - \frac{z}{A_\mu}\frac{\partial w}{\partial\alpha_\mu} \tag{2.11}$$

The effective stiffness moduli in (2.9) as coefficients are determined by the relationships

$$b_{ij}^{n\nu} = \mathbf{L}_{ijm}U_m^{n\nu} + c_{ijn\nu}, \quad b_{ij}^{*\mu\nu} = \mathbf{L}_{ijm}V_m^{\mu\nu} + zc_{ij\mu\nu} \tag{2.12}$$

where $U_m^{n\nu}(\xi, z)$, $V_m^{\mu\nu}(\xi, z)$ are periodic solutions in the variables $\xi_\mu = A_\mu y_\mu$ ($\xi = (\xi_1, \xi_2)$) with periods A_μ, respectively, for local problems in the PC /3/

$$\begin{cases} \mathbf{D}_{im}U_m^{n\nu} = -C_{in\nu} \\ N_j^{\pm}(\mathbf{L}_{ijm}U_m^{n\nu} + c_{ijn\nu}) = 0 \quad (z = z^{\pm}) \end{cases} \tag{2.13}$$

$$\begin{cases} \mathbf{D}_{im}V_m^{\mu\nu} = -c_{i3\mu\nu} - zC_{i\mu\nu} \\ N_j^{\pm}(\mathbf{L}_{ijm}V_m^{\mu\nu} + z^{\pm}c_{ij\mu\nu}) = 0 \quad (z = z^{\pm}) \end{cases}$$

Here

$$\mathbf{L}_{ijm} = \frac{1}{h}c_{ijm\nu}\frac{\partial}{\partial\xi_\nu} + c_{ijm3}\frac{\partial}{\partial z}, \quad \mathbf{D}_{im} = \frac{1}{h}\frac{\partial}{\partial\xi_\beta}\mathbf{L}_{i\beta m} + \frac{\partial}{\partial z}\mathbf{L}_{i3m}, \quad C_{im\nu} = \frac{1}{h}\frac{\partial c_{i\beta m\nu}}{\partial\xi_\beta} + \frac{\partial c_{i3m\nu}}{\partial z} \tag{2.14}$$

$$\mathbf{N}^{\pm} = \left\{\mp\frac{\partial F^{\pm}}{\partial\xi_1}, \mp\frac{\partial F^{\pm}}{\partial\xi_2}, 1\right\} \tag{2.15}$$

It can be proved that the addition of a thermal component to (1.1) will not result in a change in (2.4), but components associated with the thermal strains occur in the elasticity relationships (2.9). Relationships expressing these components in terms of the principal terms of the temperature expansion (2.2) are derived in Sect.3, while on averaged system of equations to determine these functions is obtained in Sect.4.

3. Taking into account that the layer thickness is small compared with the radii of curvature of the middle surface, we use the notation

$$k_\nu = \varepsilon k_\nu'(\alpha) \tag{3.1}$$

We set

$$\overset{0}{\alpha}_{mn} = \varepsilon\alpha_{mn}(y, z), \quad a_S^{\pm} = \varepsilon a^{\pm}(y), \quad g_S^{\pm} = \varepsilon g^{\pm}(\alpha, t, y) \tag{3.2}$$

in relationships (1.1) and (1.5).

We note that the asymptotic form (3.1), (3.2) is equivalent to neglecting the terms $k_\nu\gamma$ in deriving the thin-shell heat conduction equations /7/ and the determination of the thin plate and shell thermophysical characteristics mentioned /4, 7/.

From the Cauchy relationships /3/ and (1.1), (2.1), and (2.2) we obtain for the principal term of expansion (2.3)

$$\sigma_{ij}^{(1)} = \mathbf{L}_{ijm}u_m^{(2)} + c_{ijm\nu}\omega_{m\nu} + zc_{ij\mu\nu}\tau_{\mu\nu} - c_{ijmn}\alpha_{mn}(\theta_1^{(0)} + z\theta_2^{(0)}) \tag{3.3}$$

We substitute (3.3) into the equilibrium equations /3/, expanded in terms of ε, and conditions (1.4), and by equating the principal terms to zero we obtain

$$\mathbf{D}_{im}u_m^{(2)} = -C_{im\nu}\omega_{m\nu} - (c_{i3\mu\nu} + zC_{i\mu\nu})\tau_{\mu\nu} + \mathrm{B}_i\theta_1^{(0)} + (\beta_{i3} + z\mathrm{B}_i)\theta_2^{(0)} \tag{3.4}$$
$$N_j^{\pm}[\mathbf{L}_{ijm}u_m^{(2)} + c_{ijm\nu}\omega_{m\nu} + z^{\pm}c_{ij\mu\nu}\tau_{\mu\nu} - \beta_{ij}(\theta_1^{(0)} + z^{\pm}\theta_2^{(0)})] = 0 \quad (z = z^{\pm})$$
$$\left(\beta_{ij} = c_{ijmn}\alpha_{mn}, \quad \mathrm{B}_i = \frac{1}{h}\frac{\partial\beta_{i\nu}}{\partial\xi_\nu} + \frac{\partial\beta_{i3}}{\partial z}\right)$$

The solutions (3.4) periodic in ξ_1, ξ_2 can be represented in the form

$$u_m^{(2)} = U_m^{n\nu}\omega_{n\nu} + V_m^{\mu\nu}\tau_{\mu\nu} + S_m\theta_1^{(0)} + S_m{}^*\theta_2^{(0)} \tag{3.5}$$

where $U_m{}^{n\nu}(\xi, z)$, $V_m{}^{\mu\nu}(\xi, z)$ are solutions of the local problems (2.13), and $S_m(\xi, z)$, $S_m{}^*(\xi, z)$ are solutions periodic in ξ_1, ξ_2 for the problems

$$\begin{cases}\mathbf{D}_{im}S_m = \mathrm{B}_i \\ N_j^{\pm}(\mathbf{L}_{ijm}S_m - \beta_{ij}) = 0 \quad (z = z^{\pm})\end{cases} \tag{3.6}$$
$$\begin{cases}\mathbf{D}_{im}S_m{}^* = \beta_{i3} + z\mathrm{B}_i \\ N_j^{\pm}(\mathbf{L}_{ijm}S_m{}^* - z^{\pm}\beta_{ij}) = 0 \quad (z = z^{\pm})\end{cases}$$

We use the notation

$$s_{ij} = \beta_{ij} - \mathbf{L}_{ijm}S_m, \quad s_{ij}{}^* = z\beta_{ij} - \mathbf{L}_{ijm}S_m{}^* \tag{3.7}$$

Taking account of (3.5), (3.7) and (2.12), we obtain from (3.3)

$$\sigma_{ij}^{(1)} = b_{ij}^{\mu\nu}\omega_{\mu\nu} + b_{ij}^{*\mu\nu}\tau_{\mu\nu} - s_{ij}\theta_1^{(0)} - s_{ij}{}^*\theta_2^{(0)} \tag{3.8}$$

It is here taken into account that $b_{ij}{}^{3\nu} = 0$ /3/.
Taking the average of (3.8) according to the rule (2.7), we have

$$\langle\sigma_{ij}^{(1)}\rangle = \langle b_{ij}^{\mu\nu}\rangle\omega_{\mu\nu} + \langle b_{ij}^{*\mu\nu}\rangle\tau_{\mu\nu} - \langle s_{ij}\rangle\theta_1^{(0)} - \langle s_{ij}{}^*\rangle\theta_2^{(0)} \tag{3.9}$$
$$\langle z\sigma_{ij}^{(1)}\rangle = \langle zb_{ij}^{\mu\nu}\rangle\omega_{\mu\nu} + \langle zb_{ij}^{*\mu\nu}\rangle\tau_{\mu\nu} - \langle zs_{ij}\rangle\theta_1^{(0)} - \langle zs_{ij}{}^*\rangle\theta_2^{(0)}$$

The relationships (3.9) are a generalization of the elasticity relationships (2.9) taking thermal stresses into account. We note that $\langle s_{3j}\rangle = \langle zs_{3j}\rangle = \langle s_{3j}{}^*\rangle = \langle zs_{3j}{}^*\rangle = 0$ follows from (3.6) and (3.7). Consequently, the relationships $\langle\sigma_{3j}{}^{(1)}\rangle = \langle z\sigma_{3j}{}^{(1)}\rangle = 0$ obtained in /3/ and utilized in deriving (2.4) remain valid even when the thermal component in (1.1) is taken into account.

Substituting (3.9) into (2.4), we obtain a system of three resolving equations in the functions v_μ, w governing the principal terms of the displacement vector (2.11). The functions $\theta_\mu{}^{(0)}(\alpha, t)$ that must be determined from the solution of the heat conduction equation, will also occur in these equations.

4. Let us expand relationships (1.2) in terms of ε. Taking account of (2.2) and (3.1) we obtain

$$q_i = -\frac{\lambda_{i3}}{\varepsilon}\theta_2^{(0)} + q_i^{(0)} + \varepsilon q_i^{(1)} + O(\varepsilon^2) \tag{4.1}$$
$$q_i^{(l)} = -\lambda_{i\nu}\frac{1}{h}\frac{\partial}{\partial\xi_\nu}(\theta_1^{(l+1)} + z\theta_2^{(l+1)}) - \lambda_{i3}\frac{\partial}{\partial z}(\theta_1^{(l+1)} + z\theta_2^{(l+1)}) - \lambda_{i\nu}\frac{1}{A_\nu}\frac{\partial}{\partial\alpha_\nu}(\theta_1^{(l)} + z\theta_2^{(l)}) \quad (l = 0, 1)$$

We introduce the differential operator notation

$$\partial_1\varphi = \frac{1}{A_1A_2}\frac{\partial(A_2\varphi)}{\partial\alpha_1} \quad (1 \leftrightarrow 2) \tag{4.2}$$

Taking account of (2.2), (3.1), (4.1) and (4.2), we expand (1.3) in powers of ε^r $(r \leqslant 0)$ as follows

$$\frac{1}{h}\frac{\partial q_\nu^{(0)}}{\partial\xi_\nu} + \frac{\partial q_3^{(0)}}{\partial z} = 0 \tag{4.3}$$

$$-f + c_v\left(\frac{\partial\theta_1^{(0)}}{\partial t} + z\frac{\partial\theta_2^{(0)}}{\partial t}\right) = \frac{\lambda_{3\mu}}{\varepsilon}\partial_\mu\theta_2^{(0)} - \partial_\mu q_\mu^{(0)} + (k_1' + k_2')\lambda_{33}\theta_2^{(0)} - \frac{1}{h}\frac{\partial}{\partial\xi_\nu}\left(q_\nu^{(1)} + \lambda_{3\nu}k_\nu'z\theta_2^{(0)} - \frac{\lambda_{3\nu}}{\varepsilon^2}\theta_2^{(0)}\right) - \frac{\partial}{\partial z}\left(q_3^{(1)} - \frac{\lambda_{33}}{\varepsilon^2}\theta_2^{(0)}\right) \tag{4.4}$$

Taking account of (2.2), (3.1), (3.2) and (4.1), conditions (1.5) yield the following in expansions in powers of ε^r $(r \leqslant 1)$

$$N_i^{\pm} q_i^{(0)} = 0 \qquad (z = z^{\pm}) \tag{4.5}$$

$$N_\nu^{\pm} \left(q_\nu^{(1)} + \lambda_{3\nu} k_\nu' z^{\pm} \theta_2^{(0)} - \frac{\lambda_{3\nu}}{\varepsilon^2} \theta_2^{(0)} \right) + \tag{4.6}$$
$$N_3^{\pm} \left(q_3^{(1)} - \frac{\lambda_{33}}{\varepsilon^2} \theta_2^{(0)} \right) = \pm \omega^{\pm} [a^{\pm} (\theta_1^{(0)} + z^{\pm} \theta_2^{(0)}) - g^{\pm}] +$$
$$N_i^{\pm} \lambda_{3i} \zeta^{\pm} \theta_2^{(0)} \qquad (z = z^{\pm})$$

$$\zeta^{\pm} = \frac{z^{\pm}}{(\omega^{\pm})^2} \left[k_1' \left(\frac{\partial F^{\pm}}{\partial \xi_1} \right)^2 + k_2' \left(\frac{\partial F^{\pm}}{\partial \xi_2} \right)^2 \right] \tag{4.7}$$

where $\omega^{\pm}, N_i^{\pm}$ are defined by (2.6) and (2.15). The terms containing $\zeta^{\pm}$ are related to the expansion of the normal components $n^{\pm}$ in ε. We note that the presence of components with negative powers of ε in (4.4) and (4.6) is associated with taking account of the asymptotic form of the thin layer thermal resistance in the direction of the axis γ /4, 7/.

We introduce the notation

$$\mathbf{L}_i = \frac{\lambda_{i\nu}}{h} \frac{\partial}{\partial \xi_\nu} + \lambda_{i3} \frac{\partial}{\partial z}, \qquad \mathbf{D} = \frac{1}{h} \frac{\partial}{\partial \xi_\mu} \mathbf{L}_\mu + \frac{\partial}{\partial z} \mathbf{L}_3 \tag{4.8}$$
$$\Lambda_i = \frac{1}{h} \frac{\partial \lambda_{i\nu}}{\partial \xi_\nu} + \frac{\partial \lambda_{i3}}{\partial z}$$

We substitute the expression for $q_i^{(0)}$ from (4.1) into (4.3) and (4.5). We obtain

$$\mathbf{D}(\theta_1^{(1)} + z\theta_2^{(1)}) = -\Lambda_\mu \frac{1}{A_\mu} \frac{\partial \theta_1^{(0)}}{\partial \alpha_\mu} - (\lambda_{3\mu} + z\Lambda_\mu) \frac{1}{A_\mu} \frac{\partial \theta_2^{(0)}}{\partial \alpha_\mu} \tag{4.9}$$
$$N_i^{\pm} \left[\mathbf{L}_i (\theta_1^{(1)} + z\theta_2^{(1)}) + \frac{\lambda_{i\mu}}{A_\mu} \left(\frac{\partial \theta_1^{(0)}}{\partial \alpha_\mu} + z^{\pm} \frac{\partial \theta_2^{(0)}}{\partial \alpha_\mu} \right) \right] = 0 \quad (z = z^{\pm})$$

We represent the solution (4.9) in the form

$$\theta_1^{(1)} = W_\mu \frac{1}{A_\mu} \frac{\partial \theta_1^{(0)}}{\partial \alpha_\mu}, \qquad z\theta_2^{(1)} = W_\mu^* \frac{1}{A_\mu} \frac{\partial \theta_2^{(0)}}{\partial \alpha_\mu} \tag{4.10}$$

where $W_\mu(\xi, z)$, $W_\mu^*(\xi, z)$ are solutions periodic in ξ_ν with periods A_ν for the problems

$$\begin{cases} \mathbf{D}W_\mu = -\Lambda_\mu \\ N_i^{\pm} (\mathbf{L}_i W_\mu + \lambda_{i\mu}) = 0 \qquad (z = z^{\pm}) \end{cases} \tag{4.11}$$
$$\begin{cases} \mathbf{D}W_\mu^* = -\lambda_{3\mu} - z\Lambda_\mu \\ N_i^{\pm} (\mathbf{L}_i W_\mu^* + z^{\pm} \lambda_{i\mu}) = 0 \quad (z = z^{\pm}) \end{cases}$$

We use the notation

$$l_{i\mu}(\xi, z) = \mathbf{L}_i W_\mu + \lambda_{i\mu}, \; l_{i\mu}^*(\xi, z) = \mathbf{L}_i W_\mu^* + z\lambda_{i\mu} \tag{4.12}$$

We obtain from (4.1), (4.10) and (4.12)

$$q_i^{(0)} = -l_{i\nu} \frac{1}{A_\nu} \frac{\partial \theta_1^{(0)}}{\partial \alpha_\nu} - l_{i\nu}^* \frac{1}{A_\nu} \frac{\partial \theta_2^{(0)}}{\partial \alpha_\nu} \tag{4.13}$$

We take the average of (4.4) by using relationships (4.6) and the periodicity in y_ν

$$-\varepsilon \langle f \rangle + \varepsilon \langle c_v \rangle \frac{\partial \theta_1^{(0)}}{\partial t} + \varepsilon \langle z c_v \rangle \frac{\partial \theta_2^{(0)}}{\partial t} = \langle \lambda_{3\mu} \rangle \partial_\mu \theta_2^{(0)} - \varepsilon \partial_\mu \langle q_\mu^{(0)} \rangle - \tag{4.14}$$
$$J_0 \theta_1^{(0)} - [J_1 - (k_1 + k_2) \langle \lambda_{33} \rangle + \varepsilon Z_0] \theta_2^{(0)} + G_0$$

$$J_r = \frac{1}{V} \int_0^1 \int_0^1 [(z^+)^r \omega^+ a_S^+ + (z^-)^r \omega^- a_S^-] \, dy_1 \, dy_2 \quad (r = 0, 1, 2) \tag{4.15}$$

$$G_r = \frac{1}{V} \int_0^1 \int_0^1 [(z^+)^r \omega^+ g_S^+ + (z^-)^r \omega^- g_S^-] \, dy_1 \, dy_2 \quad (r = 0, 1)$$

$$Z_r = \frac{1}{V} \int_0^1 \int_0^1 [(z^+)^r N_i^+ \lambda_{3i}(y, z^+) \zeta^+ - (z^-)^r N_i^- \lambda_{3i}(y, z^-) \zeta^-] \, dy_1 \, dy_2$$

To obtain the second equation we take the average of the relationship (4.4) that has first been multiplied by z by using relationships (4.6) multiplied by $z^{\pm}$ and the periodicity in y_ν. Confining ourselves to a linear temperature distribution law over the layer thickness /4, 7/, we obtain

$$-\varepsilon\langle zf\rangle+\varepsilon\langle zc_v\rangle\frac{\partial\theta_1^{(0)}}{\partial t}+\varepsilon\langle z^2c_v\rangle\frac{\partial\theta_2^{(0)}}{\partial t}=\langle z\lambda_{3\mu}\rangle\partial_\mu\theta_2^{(0)}- \tag{4.16}$$
$$\varepsilon\partial_\mu\langle zq_\mu^{(0)}\rangle-J_1\theta_1^{(0)}-$$
$$\left[J_2-(k_1+k_2)\langle z\lambda_{33}\rangle+\frac{\langle\lambda_{33}\rangle}{\varepsilon}+\varepsilon Z_1\right]\theta_2^{(0)}+G_1$$

The terms $\langle q_\mu^{(0)}\rangle$, $\langle zq_\mu^{(0)}\rangle$ occur in (4.14) and (4.16), and for which

$$\langle z^rq_\mu^{(0)}\rangle=-\langle z^rl_{\mu\nu}\rangle\frac{1}{A_\nu}\frac{\partial\theta_1^{(0)}}{\partial\alpha_\nu}-\langle z^rl_{\mu\nu}^*\rangle\frac{1}{A_\nu}\frac{\partial\theta_2^{(0)}}{\partial\alpha_\nu}\quad(r=0,1) \tag{4.17}$$

follows from (4.13).

Substituting (4.17) into (4.14) and (4.16), we obtain a system of two resolving equations in the functions $\theta_\mu^{(0)}(\alpha,t)$ that determine the principal terms in the expansion of the temperature, the heat flux vector and the stresses associated with the thermal strains by means of (2.2), (4.1), (4.13) and (3.8). If the thickness changes sufficiently smoothly, then the terms Z_0, Z_1 in the equations can be neglected.

We note that as in /3/ all the effective coefficients in relationships (3.9) and (4.17) are expressed in terms of the functions $A_\mu(\alpha)$ in ξ_ν coordinates and therefore can depend on α_δ even in the case of an initially homogeneous material.

5. Taking account of relationships (3.9) and (4.17), Eqs.(2.4), (4.14) and (4.16) are equations of the quasistatic non-stationary thermoelasticity problem for an averaged shell whose effective elastic and thermophysical characteristics (the coefficients in (3.9) and (4.17)) are determined from the solution of local problems in the PC (2.13), (3.6), (4.11). All these problems are of one type and have unique solutions apart from arbitrary components periodic in ξ_ν/1, 5/. The constant components drop out upon differentiation in (2.12), (3.7), (4.12).

To formulate the boundary value problem for the equations obtained, boundary conditions must be appended on the contour Γ bounding the layer middle surface, as must an initial temperature distribution. Neglecting the boundary effect /1/, the boundary conditions on the mechanical variables can be given in the form taken in thin shell theory /7-9/ by using the principal terms of the displacement vector (2.11) and the averaged stresses (3.9).

Let us obtain the boundary and initial conditions for the heat conduction problem.

On the boundary surface of the layer Σ which is a ruled surface for which the contour Γ is a directrix while the normals to the middle surface are generators, let the following conditions be satisfied:

$$q_\mu n_\mu^\Gamma=a_\Sigma\theta-g_\Sigma(\alpha,\gamma,t)\ (\alpha\in\Gamma) \tag{5.1}$$

(a_Σ is the heat transfer coefficient, g_Σ is the external heat flux, and n_μ^Γ are components of the external unit normal to the surface Σ). In the case of a boundary condition of the third kind $g_\Sigma=a_\Sigma\theta_\Sigma$. A boundary condition of the first kind

$$\theta=\theta_\Sigma(\alpha,\gamma,t)\quad(\alpha\in\Gamma) \tag{5.2}$$

can be given on the surface Σ in place of condition (5.1).

The initial temperature distribution is

$$\theta\mid_{t=0}=\theta_0(\alpha,y,z) \tag{5.3}$$

We take the average of the relationships (5.1)-(5.3) by retaining just the principal terms of the temperature and the heat flux vector expansions. In the case of boundary conditions of the second or third kind

$$\left(-\frac{1}{\varepsilon}\langle z^r\lambda_{\mu3}\rangle\theta_2^{(0)}+\langle z^rq_\mu^{(0)}\rangle\right)n_\mu^\Gamma=\langle z^ra_\Sigma\rangle\theta_1^{(0)}+ \tag{5.4}$$
$$\langle z^{r+1}a_\Sigma\rangle\theta_2^{(0)}-\langle z^rg_\Sigma\rangle\quad(r=0,1,\alpha\in\Gamma)$$

In the case of a boundary condition of the first kind

$$\langle z^r\rangle\theta_1^{(0)}+\langle z^{r+1}\rangle\theta_2^{(0)}=\langle z^r\theta_\Sigma\rangle\quad(r=0,1,\alpha\in\Gamma) \tag{5.5}$$

The initial conditions are

$$(\langle z^r\rangle\theta_1^{(0)}+\langle z^{r+1}\rangle\theta_2^{(0)})|_{t=0}=\langle z^r\theta_0\rangle\quad(r=0,1) \tag{5.6}$$

Local problems in a PC are extended to the case of piecewise-smooth functions modelling a composite or porous material. In this case continuity conditions analogous to conditions (2.13) for the local problem presented in /3/ are added on the surfaces of discontinuity. We note that these conditions correspond to ideal contact and can be written differently /1, 5/ by taking account of the specific features of the problems being solved.

6. We examine the limit case of a "smooth" ($F^{\pm} \equiv 0$) homogeneous shell for the problem formulation under consideration. As is shown in /3/, Eq.(2.4) and relationships (2.9) and (2.10) reduce in this case to the elasticity equations and relationships taken in thin shallow shell theory /7-9/. The forces and moments are here related to the averaged stresses in the following manner (not summed with respect to β):

$$N_\beta = \varepsilon^2 \langle \sigma_{\beta\beta}^{(1)} \rangle, \quad S_{12} = \varepsilon^2 \langle \sigma_{12}^{(1)} \rangle$$
$$M_\beta = \varepsilon^3 \langle z\sigma_{\beta\beta}^{(1)} \rangle, \quad H_{12} = \varepsilon^3 \langle z\sigma_{12}^{(1)} \rangle \tag{6.1}$$

Eqs.(4.14) and (4.16) take the form

$$-\varepsilon \langle f \rangle + \varepsilon c_v \frac{\partial \theta_1^{(0)}}{\partial t} = \lambda_{3\mu} \partial_\mu \theta_2^{(0)} + \varepsilon \left(\lambda_{\mu\nu} - \frac{\lambda_{3\mu}\lambda_{3\nu}}{\lambda_{33}} \right) \partial_\mu \left(\frac{1}{A_\nu} \frac{\partial \theta_1^{(0)}}{\partial \alpha_\nu} \right) - \tag{6.2}$$
$$(a_S^+ + a_S^-)\, \theta_1^{(0)} - \left[\frac{1}{2} (a_S^+ - a_S^-) - (k_1 + k_2) \lambda_{33} \right] \theta_2^{(0)} + g_S^+ + g_S^-$$
$$-12\varepsilon \langle zf \rangle + \varepsilon c_v \frac{\partial \theta_2^{(0)}}{\partial t} = \varepsilon \left(\lambda_{\mu\nu} - \frac{\lambda_{3\mu}\lambda_{3\nu}}{\lambda_{33}} \right) \partial_\mu \left(\frac{1}{A_\nu} \frac{\partial \theta_2^{(0)}}{\partial \alpha_\nu} \right) -$$
$$6 (a_S^+ - a_S^-)\, \theta_1^{(0)} - 3 \left(a_S^+ + a_S^- + \frac{4\lambda_{33}}{\varepsilon} \right) \theta_2^{(0)} + \frac{1}{2} (g_S^+ - g_S^-)$$

In the case of convective heat transfer on the surface $S^{\pm}$, the quantities $g_s^{\pm}$ are determined from (1.6).

Relationships (6.2) are a system of heat conduction equations for an anisotropic shell under the assumption of a linear temperature distribution law over its thickness:

$$\theta = \theta_1^{(0)} (\alpha, t) + \varepsilon^{-1} \gamma \theta_2^{(0)} (\alpha, t) \tag{6.3}$$

From (6.3) we obtain expressions for the integral temperature characteristics /4, 7/

$$T = \frac{1}{\varepsilon} \int_{-\varepsilon/2}^{\varepsilon/2} \theta\, d\gamma = \theta_1^{(0)}, \qquad T^* = \frac{6}{\varepsilon^2} \int_{-\varepsilon/2}^{\varepsilon/2} \gamma\theta\, d\gamma = \frac{\theta_2^{(0)}}{2} \tag{6.4}$$

Taking account of (6.4) Eqs.(6.2) agree with analogous heat conduction equations known in two special cases: for a homogeneous isotropic shell /7/ and an anisotropic plate /4/.

Taking account of (3.2) in the isotropic case we obtain from the solution of the local problems (3.6) for coefficients (3.7) different from zero

$$s_{11} = s_{22} = \frac{1}{\varepsilon} \frac{\alpha^\theta E}{1 - \nu}, \qquad s_{11}^* = s_{22}^* = z s_{11} \tag{6.5}$$

where α^θ is the temperature coefficient of linear expansion, E is Young's modulus, and ν is Poisson's ratio. Relationships associating the thermal stresses with the integral temperature characteristics (6.4) result from (3.9), (6.1), (6.4) and (6.5) that are in agreement with those taken in thin plate and shell thermoelasticity theory /4, 7, 8/.

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